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OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Monday 18 October 2021 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

Insert

Time allowed: 2 hours
plus your additional time allowance

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INSTRUCTIONS

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INFORMATION

This Insert contains the article for Section B.

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ADDING ARCTANGENTS

Where does the name 'arctangent' come from?

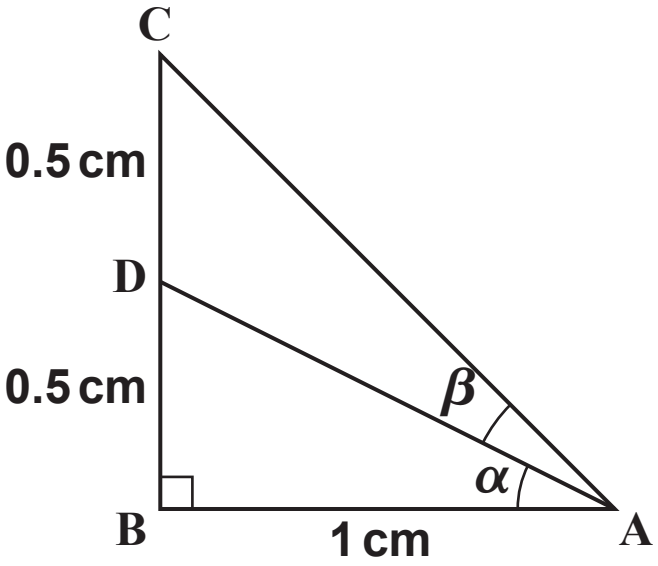
The two commonly used ways to denote the angle which has a tangent x are $\tan^{-1}x$ and $\arctan x$. The first of these is related to inverse function notation, $f^{-1}(x)$. Arctangent comes from radian measure, where an angle is represented by an arc on a unit circle; $\arctan x$ is the arc whose tangent is x.

An interesting result

It can be shown that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\arctan 1$.

Consider the diagram in FIG. C1.

FIG. C1



Triangle ABC is right-angled at B.
$AB = BC = 1$ cm.
D is the midpoint of BC.

Using triangle ABD, $\tan\alpha = \dfrac{\mathrm{DB}}{\mathrm{BA}} = \dfrac{1}{2}$ so $\alpha = \arctan\left(\dfrac{1}{2}\right)$.

Using triangle ABC, $\tan\left(\alpha+\beta\right) = 1$ so $\alpha+\beta = \arctan 1$. 15

Hence $\tan\left(\alpha+\beta\right) = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = 1$.

Using $\tan\alpha = \dfrac{1}{2}$ and finding $\tan\beta$, it follows that

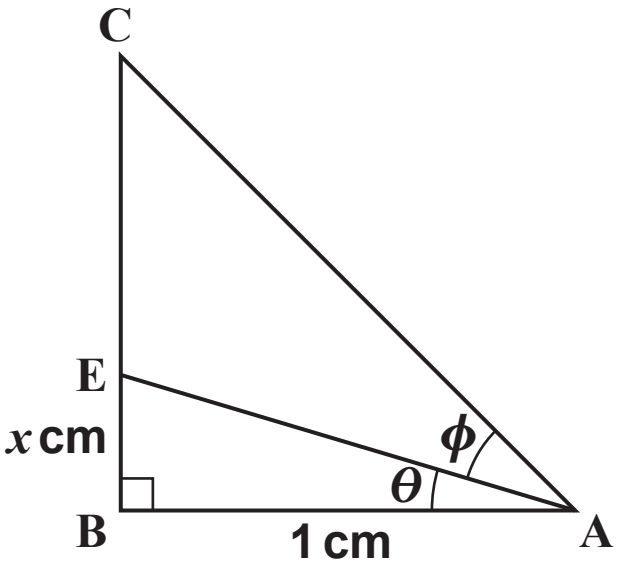
$\beta = \arctan\left(\dfrac{1}{3}\right)$,

which gives the required result that

$\arctan\left(\dfrac{1}{2}\right)+\arctan\left(\dfrac{1}{3}\right) = \arctan 1$. 20

Generalising the result

FIG. C2



Triangle ABC in FIG. C2 is the same as triangle ABC in FIG. C1 but E is a point on BC such that $EB = x$ cm and $\theta = \arctan x$.

Following the same method as above, **25**

$$\arctan x + \arctan\left(\frac{1-x}{1+x}\right) = \arctan 1.$$

The arctan addition formula

The arctangent addition formula is a further generalization:

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$, as long as $xy < 1$. 30

This result is equivalent to the addition formula $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ where $\alpha = \arctan x$ and $\beta = \arctan y$.

To see why the restriction $xy < 1$ is necessary, consider what happens if $xy \geqslant 1$. 35

Clearly, $\frac{x+y}{1-xy}$ is undefined when $xy = 1$, so the formula does not apply in this case.

Suppose next that $xy > 1$, and that x and y are both positive; in this case $y > \frac{1}{x}$.

For any positive x, $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$. 40

$y > \frac{1}{x} \Rightarrow \arctan y > \arctan\left(\frac{1}{x}\right)$ so it follows that $\arctan x + \arctan y > \frac{\pi}{2}$.

However, $\arctan\left(\frac{x+y}{1-xy}\right)$ cannot be greater than $\frac{\pi}{2}$ as the range of the arctan function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The formula $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ 45
therefore cannot be valid in this case.

A similar argument can be used to show that the formula cannot be valid when $xy > 1$ and x and y are both negative.

If $xy > 1$, the arctangent addition formula needs to 50
be adapted, as shown below.

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) - \pi$, when $xy > 1$
and $x, y < 0$

$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi$, when $xy > 1$
and $x, y > 0$ 55

Some additional results

- For n a positive integer,
  $\arctan\left(\frac{1}{n+1}\right) + \arctan\left(\frac{1}{n^2+n+1}\right) = \arctan\left(\frac{1}{n}\right)$;
  this follows directly from the arctan addition formula in line 30. 60

- $\arctan 1 + \arctan 2 + \arctan 3 = \pi$. This can be proved by using $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$
  together with $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \arctan 1$.

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